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Local instabilities to nonlinear Turing and Hopf states

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Abstract

We show a class of localized instabilities to nonlinear global states of very general form of amplitude equations. The localized instabilities are special in their spatial structure and can come in even and odd parity classes. The importance of such instabilities in the dynamics and formation of domains of a global state in 1D have been discussed.

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The study of localized spatio-temporal structures is an important field of research. The so-called topological defects in dynamical systems (e.g., chemical, hydrodynamic, etc) have attracted a lot of attention broadly in the contexts such as defect-induced transition to chaos [9–15], dynamics generated by the interaction of local and global modes [17–20], etc. Investigation of localized instabilities to global states is another important subject [1–8]. A standard way of doing weekly nonlinear analysis near an instability threshold (where such a theory actually works) is to deal with an amplitude equation of relevant form. At slow time and large spatial scales, one generally derives the amplitude equation for a slowly growing mode. An amplitude equation is of a universal form depending upon the type of instability threshold at which it has been derived [16]. To such a universal form of amplitude equations, there exist well-known global nonlinear solutions which are of Turing (spatially periodic and stationary) or Hopf types (oscillatory). In the present work, we are going to show a class of local instabilities to these global nonlinear states of amplitude equations. Thus, the results are very general and are universally applicable.

In more than one-dimensional space, the degeneracy of global states under rotation is a dominant cause of defect or localized structure formation. In an extended system, we observe domains of degenerate states interconnected by domain walls which are localized solutions of the system. In a one-dimensional system such a degeneracy of global states is missing. For an extended system in 1D, identifying the intrinsic mechanism of localized structure formation is interesting in order to understand domain formation of a global state. When localized instabilities to a global state grow, it cannot only separate regions of that state, it can also have

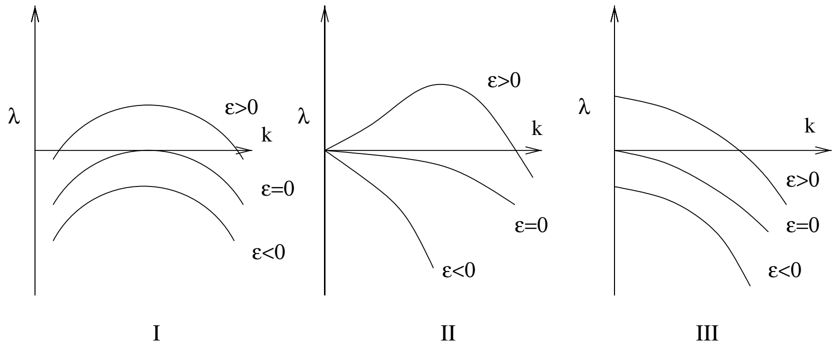

Figure 1. A schematic diagram to demonstrate several instability situations classified as I, II and III by Cross and Hohenberg. λ is the real part of growth rate and k is the wave number.

nontrivial effects on the dynamics. Internal structures of these localized instabilities can affect global phases of domains they separate. Our present work shows a special class of localized instability to the otherwise stable global solution of an amplitude equation. We will show that the localized instabilities have internal structures like quantum mechanical harmonic oscillator eigen states. Broadly, these instabilities are divided into two opposite parity classes, the even order solutions are of even parity and the odd order solutions are odd but they are generated on the same footing. In the following, we first show the form of amplitude equations and elaborate the general type of instability boundaries at which these equations are applicable, we illustrate the phase diagram of nonlinear Turing state near a type I_s instability threshold and show the instability, then we extend our result to type III_o and I_o cases and summarize in a concluding paragraph.

Near an instability boundary of type I_s—following the classification of Cross and Hohenberg [16]—the general form of an amplitude equation in one dimension is

$$\frac{\partial T}{\partial t} = \epsilon T + a\frac{\partial^2 T}{\partial x^2} - c|T|^2 T. \tag{1}$$

Coefficients a and c are functions of the actual parameters of the particular model system one considers and ϵ is the bifurcation parameter which indicates how far from the instability boundary one is placed. Since in the above form of amplitude equation we have effectively taken the slow time scale to be unity, ϵ which is otherwise dimensionless will have the dimension of inverse time. We call this amplitude equation a Turing-type amplitude equation since it is valid at an instability boundary which marks the onset of a stable Turing state. Such a form of an amplitude equation is generally investigated in connection with problems such as Rayleigh–Bénard convection, Taylor–Couette flow, electro-convection in liquid crystals, etc (see [16] and references therein). The same form of the amplitude equation is also applicable at a Hopf instability boundary (type III_o) except for the fact that the coefficients a and c are in general complex numbers. In the following part of our work, we are going to mark those in the context of a Hopf-type amplitude equation as a_h and c_h. This type of model equation is used for oscillatory chemical instabilities, nonlinear optics and laser, etc. The type I_o threshold is the situation where a coupled form of above-mentioned amplitude equations are applied to understand the interplay of spatial and temporal modes [21]. Figure 1 shows a schematic diagram to illustrate the classification of linear instability thresholds as has been done by Cross and Hohenberg [16] since we will frequently be referring to them in what follows.

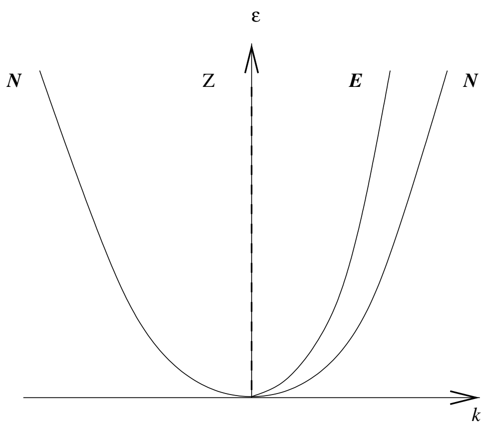

Figure 2. A schematic stability diagram of the Turing-type amplitude equation (equation (1) with $a = 1$ and $c = 1$). The nonlinear Turing state is stable inside the region bounded by the Eckhaus (E) and the zig-zag (Z) boundary.

Equation (1) has a global solution of the form $T = T_0\,\mathrm{e}^{\mathrm{i}kx}$ (we call it a nonlinear Turing state), where the wave number and the amplitude are related as $k^2 = (\epsilon - c|T_0|^2)/a$. Figure 2 shows a schematic diagram of the region of phase space where this nonlinear Turing state is stable. The diagram in figure 2 corresponds to a scaled form of the amplitude equation where $a = 1$ and $b = 1$. The zig-zag (Z) boundary at $k = 0$ is not of importance in one-dimensional analysis since it marks the onset of transverse instability to the nonlinear Turing state. The nonlinear Turing state is stable with respect to longitudinal instability inside the region between the Z and the Eckhaus (E) boundary. The E boundary is given as $k^2 = \epsilon/3$. Beyond the E boundary towards the neutral boundary (N where $k^2 = \epsilon$), one gets the other longitudinal modes growing. By longitudinal modes we mean those having wave vector parallel to the x-axis and only those are relevant in 1D. In this paper, our main purpose is to show that localized instability can grow inside the region bounded by the E and the Z boundary. The Benjamin–Feir line for type III$_o$ system (Hopf-type amplitude equation) is generally considered to be similar to the E boundary in the type I$_s$ system. We will be extending our result to analogous case of a Hopf-type amplitude equation and will show that localized instabilities can also affect the nonlinear Hopf state in a region of phase space where it is Benjamin–Feir stable.

Equation (1) has usual form of the Schroedinger equation in the absence of the nonlinear term. So, a linearized version of this equation can admit localized solutions which being bounded by a Gaussian profile are of a spatial form like Hermite polynomials. This assumption is the basis upon which we want to investigate how localized instabilities, when bounded by a Gaussian envelope, to a nonlinear Turing state behave. Perturbing the nonlinear Turing state of equation (1) by $\delta T\,\mathrm{e}^{-x^2/2b}$ we arrive at

$$\frac{\partial \delta T}{\partial t} = K\delta T + \frac{a}{b}\left[\frac{\partial^2 \delta T}{\partial x^2} - 2x\frac{\partial \delta T}{\partial x}\right], \tag{2}$$

where $K = \epsilon - a/b - cT_0^2$. Arriving at equation (2), we have neglected the term proportional to x^2/b^2. Since, the perturbation is bounded by an envelope of width $\sqrt{b}$, the relevant range of x in which we find structures is of the order of $\sqrt{b}$. So, in this range $x^2/b^2 \sim 1/b$ and it will be shown in the following that in the region of our interest $b \sim 1/k^2$ where k^2 is the wave number of the underlying Turing state. The wave number k^2 being very small b will be large which will justify the approximation. We have also rescaled the length as $x = x/\sqrt{b}$ to make it dimensionless. We have to do it since we are after spatial solutions which are generally polynomials in x.

Let us take the ansatz $\delta T = T(t)H_n(x)$ where $H_n(x)$ is the Hermite polynomial of order n. Putting this ansatz in equation (2) and separating the space and time parts we get

$$\frac{\partial^2 H_n(x)}{\partial x^2} - 2x\frac{\partial H_n(x)}{\partial x} + 2nH_n(x) = 0, \tag{3}$$

which should be associated with a temporal part giving the growth of the amplitude as

$$\frac{\partial T(t)}{\partial t} = \left(K - \frac{2na}{b}\right)T(t). \tag{4}$$

Equation (3) is the standard Hermite polynomial equation of which we will mainly discuss the zeroth- and the first-order solutions. This much is sufficient for our present purpose since we have already got an even and an odd parity localized structure on the same footing. Replacing K by its form in equation (4), we arrive at the expression of growth rate which reads

$$\lambda = a\left(k^2 - \frac{2n+1}{b}\right). \tag{5}$$

In equation (5) when a is positive, e.g. $a = 1$ as has been taken in the schematic diagram (figure 2), the localized instabilities will grow when $b > (2n+1)/k^2$. Inside the region bounded by the boundaries E and Z, k is small and that makes b a large number. Moreover, from the above expression of the growth rate we see that an odd parity (order unity) solution can dominate in linear growth over the even parity solution when its width is at least $\sqrt{3}$ times as big as the even parity (zeroth-order) solution. Thus, the relevant width of a localized instability is at least of the order of a wave length of the existing Turing pattern. At this point, the question naturally arises if higher order solutions of equation (3) are acceptable. It should be noted that $x \sim \sqrt{b}$ is a valid approximation for large enough values of n. One can expect a considerable number of orders of such localized structures to show up and grow within this region of phase space. However, higher order solutions will also require bigger spread to compete in linear growth.

At negative a, the nonlinear Turing state is unstable to such localized instabilities of width $b < 1/k^2$. It is interesting to note that at $b < 1$, x^2/b^2 is large and the approximation breaks down. When a is negative, the phase diagram shown in figure 2 does not apply. Actually, such a Turing state is every where linearly unstable to longitudinal global perturbations. However, there always exists a one-parameter family of the global nonlinear Turing solution to equation (1) and these instabilities to such solutions are interesting results.

In equation, (1) if we replace T by H and a, c by a_h, c_h, where a_h and c_h are generally complex numbers, we get the Hopf-type amplitude equation or the CGLE. Exactly, the same argument applies to this Hopf-type amplitude equation where the global solution is a travelling wave of the form $H = H_0\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$. The wave number and the frequency of the global travelling wave solution are, respectively, $k^2 = \left(\epsilon - c_{hr}H_0^2\right)/a_{hr}$ and $\omega = -a_{hi}k^2 - c_{hi}H_0^2$. Here, a_{hr}, c_{hr} are real and a_{hi}, c_{hi} are imaginary parts of a_h and c_h. Now, the real part of growth rate of the concerned localized instability to this travelling wave state is of the same form as that mentioned in equation (5). We only have to replace a by a_{hr} in equation (5) to get λ_{real}. Thus, given $\lambda_{\mathrm{real}} = a_{hr}\left(k^2 - \frac{2n+1}{b}\right)$, following the same logic as the ones mentioned in the above paragraph we can expect even as well as odd parity localized instabilities to grow for $k^2 < k_{\mathrm{BF}}^2$ where k_{BF} is the characteristic Benjamin–Feir wave number. The Benjamin–Feir line in type III_o system is analogous to the Eckhaus boundary for the type I_s case. When $k^2 > k_{BF}^2$, a resonant excitation of side bands ($q_1 + q_2 = k$) with q_1 and q_2 instability wave numbers destabilizes the nonlinear travelling wave state [16]. So, in the case of a type III_o system we also see that the Benjamin–Feir stable travelling wave state is unstable to localized instabilities of differing parity. Interestingly, the local instabilities in this case can generally be oscillatory. The emergence of an odd parity localized structure at a region inside a global travelling wave state can cause a sudden change in the phase of the travelling wave state on either side and in the immediate vicinity of the instability. Since a travelling wave will carry this local phase information at larger distances, in the long run one can expect to see domains of travelling wave state with constant phase difference. In actual experiment on chemical systems (CIMA reaction), such a situation has been observed even in the early days of chemical pattern formation [21].

Such an analysis of local instability of a coupled set of amplitude equations also holds good in a region near a type I_o instability boundary where an oscillatory spatially periodic linear instability dominates [16]. If we couple the Turing- and the Hopf-type amplitude equations as mentioned above considering coupling terms such as $|T|^2 H$ and $|H|^2 T$, we get a set of amplitude equations applicable near type I_o instability boundary. Diagonal terms in the growth equations will remain the same as in type I_s and type III_o systems. A coupling of Turing and Hops modes will appear in the off-diagonal terms. So, we can generally get localized instabilities growing under the same conditions mentioned for the type I_s and type III_o cases. Due to coupling to Hopf modes, the instabilities can also be oscillatory for the Turing-type solution and generate some interesting dynamics.

To conclude, we like to mention that our present work describes a very important class of localized instabilities. These instabilities are investigated on a very general basis of amplitude equations which are applicable to any system having linear instability boundaries of types I_s, III_o and I_o. The universal form of amplitude equations employed are often taken as model systems to investigate various nonlinear phenomenon. These are the general situations where one can consider ϵ as a large number, a negative, etc. Thus, the results are not only important for systems close to linear threshold but are also generally applicable to the global nonlinear states. We have shown in 1D that a nonlinear Turing state is unstable to localized instabilities inside the phase space bounded by the Eckhaus and zig-zag boundaries. This is the region of phase space where the Turing state is stable with respect to longitudinal and transverse instabilities. The present analysis suggests the intrinsic mechanism of the formation of domains of a single Turing state in extended 1D systems. The same analysis is repeated for the type III_o case where localized instabilities can separate domains of a global travelling wave state. In analogy with the type I_s system, it has been pointed out that the instability grows in a region where the travelling wave is Benjamin–Feir stable. The instabilities shown have a width of the order of wave length of the underlying Turing or travelling wave states and cannot grow in a homogeneous oscillatory background. It is important to note that the type III_o amplitude equation has a homogeneous oscillatory global solution which is never unstable to such localized instabilities. The instabilities essentially require an underlying length scale to be selected. A very special fact about these instabilities is that they can appear in even and odd parity varieties. Such an odd parity localized structure can invert the phase of the underlying states over a length scale (λ) at which the phase is supposed to get restored. We have also given arguments in order to extend the results to type I_o situation where a coupled set of amplitude equations are employed. In this latter case, such instabilities can considerably add to the complexity of the system by making the Turing state unstable to oscillatory localized structures. Finally, we would like to mention that although the present analysis is in 1D the result can easily be extended in two dimensions.

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